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Long-range stiffness of spectral fluctuations in integrable scale-invariant systems

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Abstract. In this paper we study the semiclassical limit for the Δ_3 statistic of integrable systems which have a homogeneous polynomial as potential. These systems possess a scale invariance which provides us with the energy dependence of the statistic using the general expressions given by Berry. We also obtain the functional dependence of the universal part of the Δ_3 statistic both for integrable and ergodic systems. We use a stationary phase approximation to evaluate the semiclassical limit of the level density when the dimension of the system is larger than one. The investigation of its validity leads to the introduction of a generalised perimeter term in the average level density. The fluctuating part of the semiclassical level density yields the semiclassical limit of the Δ_3 statistic, which is compared numerically to results obtained for actual spectra. We find that the semiclassical approximation is excellent provided the perimeter term is taken into account exactly. We also study the dimensional dependence of Δ_3. For energy levels around the Nth level above the ground state the position of the kink in the Δ_3 statistic is essentially proportional to $N^{(d-1)/d}$.

1. Introduction

Recently quantum spectra of classically chaotic systems have received a considerable amount of attention (see Bohigas and Giannoni (1984) for a review). Both analytical and numerical evidence has been found that the spectral fluctuations of completely chaotic systems can be described by the invariant random matrix ensembles. Most recently, Berry (1985) was able to give a semiclassical derivation of the correct logarithmic dependence of the Δ_3 statistic. This statistic is a function of L, the length of an interval along the spectrum in units of average level spacings. It measures the fluctuations of the number of quantum levels in intervals of length L. The transition region between completely chaotic and integrable systems has been subjected to several studies (Yukawa 1985, Ishikawa and Yukawa 1985, Seligman *et al* 1984, 1985) but has not yet been fully understood. However, there is also evidence that in this case random matrix theory plays an important role (Seligman *et al* 1985). In this context a problem arose as, close to the integrable limit, the non-universal properties of the level correlation function are very pronounced. In particular we refer to the fact that the $\Delta_3(L)$ statistic no longer saturates after a certain value of L. This so-called 'kink' was first reported by Zirnbauer and the present authors (Seligman *et al* 1984). After that it was studied by Casati *et al* (1985) for the square well. In the same study they showed that even the nearest-neighbour spacing distribution of their Hamiltonian is non-generic (see also Seligman and Verbaarschot 1986a).

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Recently, Berry (1985) explained the phenomenon of the 'kink' by using the semiclassical limit of the fluctuating part of the level density. He was able to give a quantitative explanation of the data of Casati *et al* (1985). However, his theory has not been tested for systems other than the square well. It is the objective of this paper to perform this task. We will concentrate on the class of separable homogeneous potentials. This class has the advantage that all calculations can be performed explicitly. We take this opportunity to make a careful analysis of the relevant stationary phase approximations.

Since actual physical systems usually involve many degrees of freedom we have also been interested in the dimensional dependence of the non-universal properties of the Δ_3 statistic. Intuitively it is clear that the position of the 'kink' will move to larger values of L when we increase the dimensionality of the system. We shall also investigate this surmise for the aforementioned systems.

The organisation of this paper is as follows. In § 2 we define the model and use general scaling arguments to derive the energy dependence of the Δ_3 statistic. In § 3 we evaluate the semiclassical limit of the fluctuating part of the level density. The stationary phase approximation is discussed and a generalised perimeter correction to the average level density is derived. In § 4 we derive an explicit expression for the Δ_3 statistic. Numerical results are presented in § 5 and concluding remarks are made in § 6.

2. Scaling properties of the Δ_3 statistic

We consider integrable systems with a homogeneous polynomial as potential. In a d-dimensional space the Hamiltonian is given by

$$H = \tfrac{1}{2}\sum_{i=1}^{d} p_i^2 + V^{(2m)}(x_1, \ldots, x_d) \tag{2.1a}$$

where

$$V^{(2m)}(x_1, \ldots, x_d) = \sum_{i=1}^{d} c_i x_i^{2m}. \tag{2.1b}$$

Hamiltonians of the form (2.1) have scaling properties. Solutions of the Hamilton equations of motion at different energies are related by a simple scale transformation of the coordinates and the time (Landau and Lifshitz 1967). For the WKB approximation to the eigenvalues E this implies that

$$E = \sum_{i=1}^{d} \gamma_i (n_i + \alpha_i)^p \hbar^p \qquad p = 2m/(m-1). \tag{2.2}$$

Here, the n_i are integers larger than or equal to zero and the γ_i and α_i are constants to be discussed below. Equation (2.2) follows immediately by expressing the Hamiltonian in action-angle variables. The actions along one period of a periodic trajectory are

$$I_i = \frac{1}{2\pi}\oint p_i \, \mathrm{d}x_i = a_i E_i^{1/2+1/2m}. \tag{2.3a}$$

The constants a_i are given by

$$a_i = \frac{\sqrt{2}}{\pi m} c_i^{-1/2m} B\left(\frac{3}{2}, \frac{1}{2m}\right) \tag{2.3b}$$

where $B(x, y)$ is the beta function. In terms of the variables I_i the Hamiltonian can

be written as

$$H = \sum_{i=1}^{d} \left(\frac{I_i}{a_i}\right)^{2m/(m+1)} \tag{2.4}$$

In (2.2) the values of the α_i are given by

$$\begin{aligned} &\alpha_i = 0.5 && \text{for } 1 < p < \infty \\ &\alpha_i = 0 && \text{for } p = \infty \text{ (von Neumann boundary conditions)} \\ &\alpha_i = 1.0 && \text{for } p = \infty \text{ (Dirichlet boundary conditions).} \end{aligned} \tag{2.5}$$

From (2.4) one immediately deduces the WKB approximation given in (2.2) (the constants γ_i are given by $\gamma_i = a_i^{-p}$). At this point we want to remark that for Hamiltonians with a discrete symmetry the value of α_i is different from (2.5) when we consider only levels of one symmetry class. For instance, the Hamiltonian (2.1) is invariant under the parity operation. The odd and even parity states will be considered separately. In a one-dimensional system the even parity levels are given by

$$E_n = \gamma(2n+\tfrac{1}{2})^p = \gamma 2^p (n+\tfrac{1}{4})^p \tag{2.6}$$

and the odd parity levels by

$$E_n = \gamma(2n+1+\tfrac{1}{2})^p = \gamma 2^p (n+\tfrac{3}{4})^p. \tag{2.7}$$

We have used the fact that in a one-dimensional potential of the form (2.1) the levels are ordered according to the number of nodes in the wavefunction. The value of α_i is of importance for the discussion in § 3.

The accuracy of the WKB approximation has been studied numerically by diagonalising Hamiltonians of the form (2.1) in a harmonic oscillator basis. As an illustration we show in the second column of table 1 some eigenvalues corresponding to positive parity states of an x^4 potential (the index of the eigenvalues is given in the first column). In the third column we give the WKB approximation calculated from $c(n+\frac{1}{4})^{4/3}$ ($c = 1.172\,034\,0504$). The agreement is very good. Analytical estimates have been given by Hioe *et al* (1978). They found that the deviation from the WKB result is of order n^{-2} in the one-dimensional case. Therefore, to leading order in n^{-1} these deviations do not contribute to the correlations of the eigenvalues in higher-dimensional systems. This will become clear in § 3. In conclusion, we can use (2.2) as a starting point for the study of correlations of eigenvalues of Hamiltonians (2.1).

We will restrict our study to correlations that extend over a large but in the limit $\hbar \to 0$ finite number of eigenvalues. The appropriate measure for these correlations is

Table 1. Comparison of exact eigenvalues and their WKB approximation for a one-dimensional x^4 potential.

n	Exact value	WKB value
0	0.185	0.226
1	1.578	1.587
2	3.456	3.461
5	10.694	10.697
10	26.096	26.098
25	86.820	86.822
50	217.331	217.332
100	538.577	538.577

the Δ_3 statistic defined by

$$\Delta_3(L) = \min_{A,B} \frac{\bar{\rho}}{L} \int_{-L/2\bar{\rho}}^{L/2\bar{\rho}} d\varepsilon (N(E+\varepsilon) - A - B\varepsilon)^2. \tag{2.8}$$

Here, $N(E)$ is the integrated level density (a staircase function) and $\bar{\rho}$ is the average level density. The minimisation is over A and B. For the evaluation of Δ_3 we will use the semiclassical approximation to the level density with inclusion of the lowest order quantum fluctuations. According to the work of Gutzwiller (1967) $\rho(\varepsilon)$ is given by (see also Berry and Mount 1972)

$$\rho(\varepsilon) = h^{-d} \int \delta(E-H)\, d^d x\, d^d p + \hbar^{-(d+1)/2} \sum_j A_j(E) \exp(iS_j(E)/\hbar) \tag{2.9a}$$

for integrable systems and

$$\rho(\varepsilon) = h^{-d} \int \delta(E-H)\, d^d x\, d^d p + \hbar^{-1} \sum_j A_j(E) \exp(iS_j(E)/\hbar) \tag{2.9b}$$

for ergodic systems. The sum is over all closed periodic trajectories with energy E. $S_j(E)$ is the action along one such trajectory and $A_j(E)$ are coefficients that can be derived from the properties of the classical trajectories. Note that the physical dimension of the amplitudes $A_j(E)$ is different in the integrable and ergodic cases. Since a number theoretic derivation of (2.9*a*) will be given in § 3, we do not quote the explicit expressions for $S_j(E)$ and $A_j(E)$. For the interested reader we refer to the work of Gutzwiller (1967, 1970, 1971) and Berry and Tabor (1977). By using the scaling properties of the Hamiltonian (2.1) we will derive a result for the energy dependence of $\Delta_3(L)$. For the study of the eigenvalue correlations in an interval around E containing a finite number of eigenvalues we can make the following linear approximation to the action:

$$S_j(E+\varepsilon) = S_j(E) + \varepsilon\, dS_j/dE. \tag{2.10}$$

In the limit $h \to 0$ this approximation is exact. In this limit the energy dependence of the smooth functions $A_j(E)$ is irrelevant (see Berry 1985). By changing to dimensionless integration variables we can write $\Delta_3(L)$ as

$$\Delta_3(L) = \min_{A,B} \int_{-1}^{1} d\varepsilon \left[N\left(E + \frac{\varepsilon L}{2\bar{\rho}}\right) - A - B\varepsilon \right]^2. \tag{2.11}$$

Due to the minimisation the constant multiplying B can be omitted. Only the oscillating part of $N(E)$ contributes to Δ_3. Since ε is dimensionless $\Delta_3(L)$ is a function of $(L/2\bar{\rho}\hbar)\, dS_j/dE$. For scaling systems the energy dependence can be derived from dimensional arguments only. For the relevant quantities we find the following dependence on $\hbar$:

$$\frac{L}{2\bar{\rho}\hbar} \frac{dS_j}{dE} \sim [S]^{1-d} \hbar^{d-1} \tag{2.12a}$$

$$\hbar^{1-d} \left(\frac{dS_j}{dE}\right)^{-2} A_j^2 \sim [S]^{d-1} \hbar^{1-d} \qquad \text{for integrable systems} \tag{2.12b}$$

$$\hbar^{0} \left(\frac{dS_j}{dE}\right)^{-2} A_j^2 \sim [S]^{0} \hbar^{0} \qquad \text{for ergodic systems} \tag{2.12c}$$

$$N(E) \sim [S]^d \hbar^{-d}. \tag{2.12d}$$

[S] denotes the dimension of an action. Putting everything together results in

$$\Delta_3(L) = N(E)^{(d-1)/d} f(L(N(E))^{(1-d)/d}) \tag{2.13a}$$

for integrable systems and

$$\Delta_3(L) = g(L(N(E))^{(1-d)/d}) \tag{2.13b}$$

for ergodic systems, where f and g are functions that do not depend on $\hbar$, L and $N(E)$. This energy dependence in (2.13b) is consistent with the saturation value of the Δ_3 statistic derived by Casati *et al* (1985) for the square well. The generalisation of their argument to an arbitrary scaling potential yields the same asymptotic dependence. From (2.13a) we also conclude that the universal part of the Δ_3 statistic, i.e. the part that does not depend on $N(E)$, has to be linear in the integrable case. In the ergodic case we consider the difference $\Delta_3(L) - \Delta_3(1)$. It is clear that the universal part of this statistic can only have a logarithmic dependence on L. Berry (1985) derived the asymptotic limit of Δ_3 by using a sum rule for the amplitudes A_j derived by Hannay and Ozorio de Almeida (1984). Up to constants this sum rule can be derived by scaling arguments for systems with homogeneous polynomials as a potential (Seligman and Verbaarschot 1986b).

From (2.12) it is clear that an expansion in powers of $\hbar$ is also an expansion in inverse powers of $N(E)$.

3. Semiclassical limit of the level density

In § 2 we have explained that the eigenvalues of Hamiltonians of the form (2.1) are given by the WKB approximation (2.2). Berry and Tabor (1976, 1977) have derived general formulae for the semiclassical level density of integrable systems. Since for the eigenvalues (2.2) all calculations can be carried out explicitly without loss of the essential features of the general case we present a derivation of the semiclassical level density. For simplicity we consider the two-dimensional case. The generalisation to arbitrary dimensions is immediate.

The level density following from (2.2) is given by (from now on we put $\hbar = 1$)

$$\rho(\varepsilon) = \sum_{n_i=0}^{\infty} \delta[E - \gamma_1(n_1+\alpha_1)^p - \gamma_2(n_2+\alpha_2)^p]. \tag{3.1}$$

We use the Poisson summation formula to write the level density as (see Berry and Tabor 1976)

$$\rho(E) = \int_0^\infty \int_0^\infty \mathrm{d}n_1\, \mathrm{d}n_2 \sum_{M_1 M_2} \exp\{2\pi \mathrm{i}[M_1(n_1-\alpha_1) + M_2(n_2-\alpha_2)]\}\, \delta(E - \gamma_1 n_1^p - \gamma_2 n_2^p). \tag{3.2}$$

The δ function can be evaluated by transforming the n_i variables to 'polar' coordinates

$$n_1^p \gamma_1 = r\cos^2\varphi \qquad n_2^p \gamma_2 = r\sin^2\varphi. \tag{3.3}$$

The Jacobian J of this transformation is given by

$$J = 2(\gamma_1\gamma_2)^{-1/p} p^{-2} (r\cos\varphi\sin\varphi)^{2/p-1}. \tag{3.4}$$

We note that for $0<p<2$ this Jacobian is zero for φ equal to zero and $\pi/2$. The r integration can be carried out. This yields for the level density

$$\rho(E)=\frac{2}{p^2}E^{2/p-1}(\gamma_1\gamma_2)^{-1/p}\int_0^{\pi/2}\mathrm{d}\varphi\sum_{\substack{M_1\\M_2}}\left(\frac{\sin 2\varphi}{2}\right)^{2/p-1}$$
$$\times\exp\left\{2\pi\mathrm{i}\left[M_1\left(\frac{E}{\gamma_1}\right)^{1/p}\cos^{2/p}\varphi+M_2\left(\frac{E}{\gamma_2}\right)^{1/p}\sin^{2/p}\varphi-\alpha_1M_1-\alpha_2M_2\right]\right\}. \tag{3.5}$$

When both M_1 and M_2 are different from zero the φ integration can be performed by the method of stationary phases. In the limit $E\to\infty$ the results thus obtained become exact. When either M_1 or M_2 equals zero the stationary point is at the boundary of the integration manifold. Just at these points the Jacobian J in (3.4) is equal to zero for $0<p<2$. Moreover, for $0<p<2$ there are two stationary points, one at zero and the other at $\pi/2$. The second derivative with respect to φ at one of these points is infinite for $1<p<2$. (For billiard problems ($p=2$), where most of these problems do not occur, the stationary points at the boundaries have been discussed by Richens and Berry (1981).) This suggests that to make a careful analysis of this case we should not transform the n_i variables to 'polar' coordinates but instead use (3.2) as a starting point. We will add the terms with either M_1 or M_2 equal to zero to the term with both M_1 and M_2 equal to zero. Together they define the average level density $\rho_{\mathrm{SM}}(E)$. We will evaluate these terms exactly. Thereby we circumvent the problems involving the stationary phase approximation. For ρ_{SM} we obtain

$$\rho_{\mathrm{SM}}(E)=\int_0^\infty\int_0^\infty \mathrm{d}n_1\,\mathrm{d}n_2\Bigg(1+\sum_{M_1\neq0}\exp[2\pi\mathrm{i}M_1(n_1-\alpha_1)]$$
$$+\sum_{M_2\neq0}\exp[2\pi\mathrm{i}M_2(n_2-\alpha_2)]\Bigg)\delta(E-\gamma_1n_1^p-\gamma_2n_2^p)$$
$$=\int_0^\infty\int_0^\infty \mathrm{d}n_1\,\mathrm{d}n_2\Bigg(\sum_{k_1=0}^{\infty}\delta(n_1-\alpha_1-k_1)$$
$$+\sum_{k_2=0}^{\infty}\delta(n_2-\alpha_2-k_2)-1\Bigg)\delta(E-\gamma_1n_1^p-\gamma_2n_2^p). \tag{3.6}$$

The integrations can be carried out easily. This yields

$$\rho_{\mathrm{SM}}(E)=\frac{1}{p}\gamma_2^{-1/p}\sum_{\substack{k_1=0\\ \gamma_1(k_1+\alpha_1)^p<E}}[E-\gamma_1(k_1+\alpha_1)^p]^{1/p-1}+\frac{1}{p}\gamma_1^{-1/p}$$
$$\times\sum_{\substack{k_2=0\\ \gamma_2(k_2+\alpha_2)^p<E}}[E-\gamma_2(k_2+\alpha_2)^p]^{1/p-1}-\frac{1}{p^2}(\gamma_1\gamma_2)^{-1/p}E^{2/p-1}B\left(\frac{1}{p},\frac{1}{p}\right) \tag{3.7}$$

where $B(x,y)$ is the beta function. When $E\to\infty$ $\rho_{\mathrm{SM}}(E)$ tends to the Thomas-Fermi approximation for the average level density. However, for the calculation of the correlation function we have to keep all terms in (3.7). We evaluate the corrections in leading order in $E^{-1/p}$. Since the first two terms in (3.7) are both Riemann sums

that approximate the integral given by the last term, in lowest order in $E^{-1/p}$ the corrections to the leading order term $p^{-2}(\gamma_1\gamma_2)^{-1/p}E^{2/p-1}B(1/p, 1/p)$ are given by

$$\frac{1}{2p}\gamma_2^{-1/p}(\tfrac{1}{2}-\alpha_1)E^{1/p-1}+\frac{1}{2p}\gamma_1^{-1/p}(\tfrac{1}{2}-\alpha_2)E^{1/p-1}. \tag{3.8}$$

By inspection of the potential (2.1) one immediately concludes that for $p=2$, (3.8) yields the perimeter term of the semiclassical level density of a square box (see Richens and Berry 1981, Bohigas and Giannoni 1984). One also observes that this term changes sign going from von Neumann boundary conditions ($\alpha_i=0$) to Dirichlet boundary conditions ($\alpha_i=1$). When the levels are given by the WKB approximation ($\alpha_i=0.5$) there is no perimeter term. However, when one considers levels of a definite symmetry class the value of α_i is no longer 0.5 (see (2.5)) and one has to take into account the term given in (3.8). Actually, the higher-order terms in (3.7) belong to the fluctuating part of the spectrum. It is because of the problems in calculating these terms by a stationary phase approximation that we interpret $\rho_{SM}(E)$ as the average level density.

The fluctuating part of the level density is given by the terms in (3.5) for which neither M_1 nor M_2 is equal to zero. The φ integration will be performed by the method of stationary phases. In this case there are no additional problems. The stationary point is given by

$$M_1\gamma_1^{-1/p}\cos^{2/p-2}\varphi = M_2\gamma_2^{-1/p}\sin^{2/p-2}\varphi \tag{3.9}$$

and the second derivative of the exponent at this point is equal to

$$2\pi i E^{1/p}\frac{4(1-p)}{p^2}[(M_1\gamma_1^{-1/p})^{p/(p-1)}+(M_2\gamma_2^{-1/p})^{p/(p-1)}]^{(p-1)/p}. \tag{3.10}$$

Using these results we obtain the following stationary phase approximation for the fluctuating part of the level density:

$$\rho_{osc}(E)=\frac{1}{p(p-1)^{1/2}}E^{3/2p-1}\sum_{\substack{M_1M_2\neq 0\\ \mathrm{sgn}(M_1)=\mathrm{sgn}(M_2)}}\frac{(\gamma_1\gamma_2)^{1/2(1-p)}(M_1M_2)^{(2-p)/2(p-1)}}{[(M_1/\gamma_1^{1/p})^{p/(p-1)}+(M_2/\gamma_2^{1/p})^{p/(p-1)}]^{(3-p)/2p}}$$
$$\times\exp\{2\pi i E^{1/p}[(M_1/\gamma_1^{1/p})^{p/(p-1)}+(M_2/\gamma_2^{1/p})^{p/(p-1)}]^{(p-1)/p}-\tfrac{1}{4}\pi i\,\mathrm{sgn}\,M_1\}. \tag{3.11}$$

This result will be the starting point of the calculation of the Δ_3 statistic.

We do not give the derivation for the d-dimensional case here but just quote the final result:

$$\rho_{osc}(E)=\sum_{M_i\neq 0}A_M\exp(iS_M)$$
$$A_M=\left(\prod_i\gamma_i\right)^{1/2(1-p)}\left(\prod_i M_i\right)^{(2-p)/2(p-1)}E^{(d+1)/2p-1}\frac{(p-1)^{1/2}}{p}(p-1)^{-d/2}$$
$$\times\left[\sum_{i=1}^{d}\left(\frac{M_i}{\gamma_i^{1/p}}\right)^{p/(p-1)}\right]^{[-(p-1)/2p][d/(p-1)-1]}\exp(\tfrac{1}{4}\pi i\beta)$$
$$S_M=2\pi E^{1/p}\left[\sum_{i=1}^{d}\left(\frac{M_i}{\gamma_i^{1/p}}\right)^{p/(p-1)}\right]^{(p-1)/p}. \tag{3.12}$$

(For the definition of β see Berry and Tabor (1976).) For the interpretation of this equation in terms of periodic trajectories we refer to Berry (1985).

The average number of levels $N(E)$ below energy E is obtained by integrating ρ_{SM} up to E. We will express the energy E in (3.11) and (3.12) in terms of $N(E)$. In doing this the additional terms given by (3.7) give rise to contributions that vanish in leading order in $E^{-1/p}$. Therefore we only use the Thomas-Fermi approximation $\bar{\rho}$ to the level density to express E in $N(E)$. In d dimensions $\bar{\rho}$ is given by

$$\bar{\rho}_d(E) = E^{d/p-1}\left(\prod_i \gamma_i\right)^{-1/p} p^{-d} I_{d,1/p-1,1/p-1}$$
$$I_{d,r,s} = \int \mathrm{d}E_1 \ldots \mathrm{d}E_d\, \delta(E_1 + \ldots + E_d - 1)\left(\prod_{i=2}^{d} E_i\right)^r E_1^s. \tag{3.13}$$

The integral $I_{d,r,s}$ can be solved recursively:

$$I_{d,r,s} = B(r+1, s+1) I_{d-1,r+s+1,s+1}. \tag{3.14}$$

4. Evaluation of the Δ_3 statistic

In this section we evaluate the semiclassical limit of the Δ_3 statistic for the class of potentials defined in (2.2). Our starting point is the expression for Δ_3 as given by Berry (1985)

$$\Delta_3(L) = 2\sum_{M}\left(\frac{A_M}{T_M}\right)^2 G\left(\frac{LT_M}{2\bar{\rho}}\right). \tag{4.1}$$

The coefficients A_M are given in (3.12) and T_M is defined by

$$T_M = \mathrm{d}S_M/\mathrm{d}E. \tag{4.2}$$

The function G is specific for the correlations described by Δ_3 and is given by

$$G(y) = 1 - \frac{\sin^2 y}{y^2} - \frac{3(y\cos y - \sin y)^2}{y^4}. \tag{4.3}$$

After insertion of the formulae for A_M and T_M in (4.1) and expressing the energy E in $N(E)$ we obtain for $\Delta_3(L)$

$$\Delta_3(L) = \frac{2^{d-2}}{\pi^2}\left(\frac{p}{2(p-1)}\right)^{d-1}\left(\frac{d}{pI_{d,1/p-1,1/p-1}}\right)^{(d-1)/d}\left(\prod_i \gamma_i\right)^{-1/(p-1)+(d-1)/pd}(N(E))^{(d-1)/d}$$
$$\times \sum_{M_i>0}\left(\prod_i M_i\right)^{(2-p)/(p-1)}\left[\sum_{i=1}^{d}\left(\frac{M_i}{\gamma_i^{1/p}}\right)^{p/(p-1)}\right]^{[(p-1)/p][d/(1-p)-1]}$$
$$\times G\left(L\pi(N(E))^{(1-d)/d}\left(\prod_i \gamma_i\right)^{1/pd}\right.$$
$$\left.\times\left(\frac{p}{d}\right)^{(d-1)/d} I_{d,1/p-1,1/p-1}^{-1/d}\left[\sum_{i=1}^{d}\left(\frac{M_i}{\gamma_i^{1/p}}\right)^{p/(p-1)}\right]^{(p-1)/p}\right). \tag{4.4}$$

For $d = 1$ there are no fluctuating contributions to the level density and the Δ_3 statistic is equal to its minimum value ($\frac{1}{12}$). For $d = 2$ and $p = 2$ (4.4) reduces to the expression for the square well given by Berry (1985). However, there is one difference. We have included the terms with either M_1 or M_2 equal to zero in the average level density. Up to the order in $E^{-1/p}$ we are considering, the result of Berry (1985) differs because

we have treated the perimeter term exactly. Berry evaluated the fluctuating part of the perimeter term via a stationary phase approximation. The stationary points lie on the boundary of the integration manifold. This yields an additional factor of 0.5 for A_M and a factor 0.25 for A_M^2.

For large values of y the function $G(y) = 1$. Therefore, the value of $\Delta_3(L)$ saturates when $L \to \infty$. From (4.4) it follows that the energy dependence of the saturation value is given by

$$\Delta_3(\infty) \sim (N(E))^{d/(d-1)}. \tag{4.5}$$

This result was already derived in § 2 by using scaling arguments. For the square well it has been given by Casati *et al* (1985).

5. Numerical results

In this section we discuss the numerical results. We calculate the Δ_3 statistic for a sequence of eigenvalues generated by the WKB approximation given in (2.2) and compare it to the analytical result given in (4.4). In the case of the WKB eigenvalues we unfold the spectrum by means of the semiclassical level density (3.7) (i.e. the spectrum is transformed to a spectrum with average level spacing equal to one but with the same level fluctuations). We study three different potentials of the form (2.1), the homogeneous x^4 polynomial, the homogeneous x^6 polynomial and the square well. We consider different energies and different values of the parameters γ_i. In the case of the x^4 and x^6 potentials the value of α_i is taken to be equal to 0.25 (even parity states). For the square well we take $\alpha_i = 1.0$.

In figure 1 we present results for the 500th eigenvalue up to the 1500th eigenvalue. The dots represent the values of the Δ_3 statistic for the WKB eigenvalues and the full curve corresponds to the analytical expression (4.4) calculated for the 1000th eigenvalue. Going from the top to the bottom the ratio γ_1/γ_2 is equal to $\frac{1}{2}(\sqrt{5}+1)$, π, $\Gamma(\frac{1}{4})$. The error in the Δ_3 statistic is calculated from its variation over the spectrum. For small values of L the error is much smaller than for large values of L. The reason is twofold. In the first place the error is reduced by the spectral averaging. In the second place, for large values of L the error has an important systematic component due to the fact that the saturation value of Δ_3 changes by a factor $\sqrt{3}$ going from the 500th to the 1000th eigenvalue. Since the saturation value of Δ_3 is approximately symmetric around the centre of the part of the spectrum under consideration the actual error is less than the error due to the non-stationarity. The large deviation in figure 1(*a*) (x^4) is a consquence only of the large deviations for the 1000th eigenvalue up to the 1500th eigenvalue. For the 500th eigenvalue up to the 1000th eigenvalue and for the 1500th eigenvalue up the 2000th eigenvalue we do not find any deviations.

In figure 2 we present results for the 4500th eigenvalue up to the 5500th eigenvalue. The potentials (also the values of γ_i and α_i) are the same as in figure 1. Now the saturation value of Δ_3 only changes by a factor $(11/9)^{1/2}$. Therefore our errors are mainly of statistical origin. Indeed, we find that the fluctuations of the dots around the full curve are of the order of the error bars. Moreover the errors are slightly smaller that in the $E = 1000$ case (note that the scale on the Y axis differs by a factor 2). All curves in figures 1 and 2 look very similar. However, when we take the ratio of γ_1 and γ_2 very different from 1 we find different shapes for the Δ_3 statistic. Since this degenerate case is of little physical interest we will not discuss it any further.

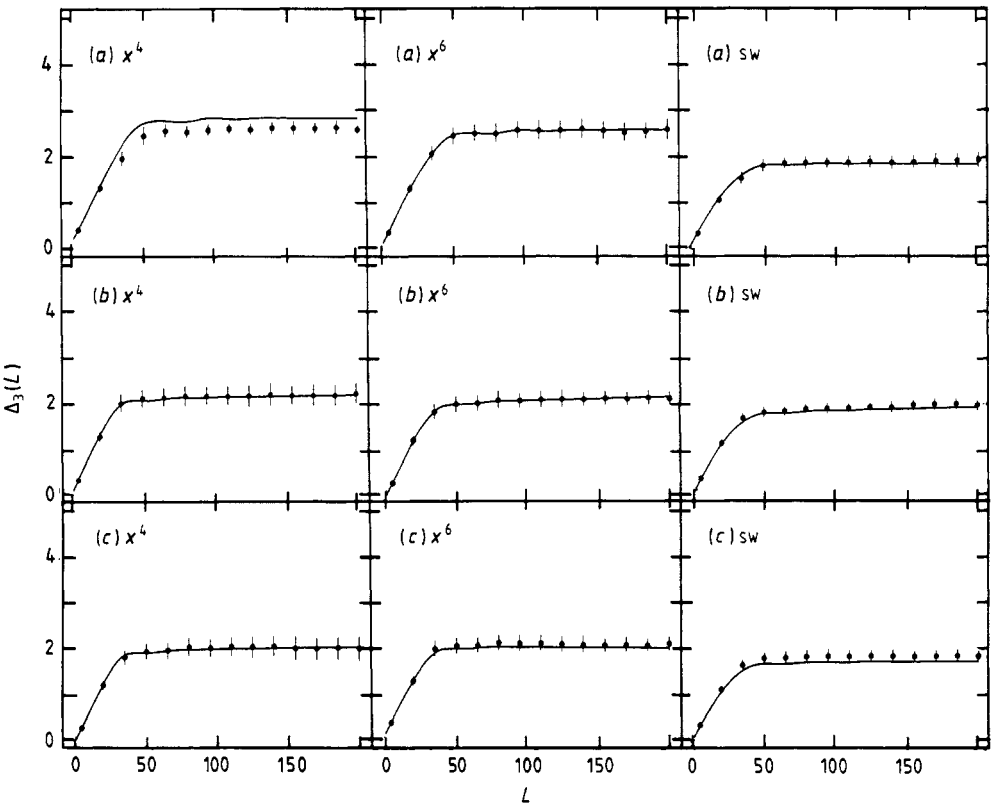

Figure 1. Comparison of the Δ_3 statistic calculated from the WKB eigenvalues (dots with error bars) and the analytical expression given in (4.4) (full curve) for the 500th eigenvalue up to the 1500th eigenvalue. The form of the potential is denoted in the figure (SW refers to square well). The ratio of γ_1/γ_2 in (a), (b), (c) is given by $\frac{1}{2}(\sqrt{5}+1)$, π, $\Gamma(\frac{1}{4})$, respectively.

In figure 3 we show the effect of not taking into account the fluctuating part of the perimeter term in the average level density used for unfolding the spectrum. The parameters of the potential are equal to those of figure $1(b)$. In figure $3(a)$ the values of α_1 and α_2 are equal to 0.25 and in figure $3(b)$ α_1 is equal to 0.25 and α_2 is equal to 0.75. There are much larger deviations in the asymptotic region than in the corresponding figure in figure 1.

The dimensional dependence of Δ_3 is shown in figure 4. We consider the case of a homogeneous x^4 potential and calculate the expression given in (4.4) for $d=2$, $d=3$ and $d=4$. The constants γ_i are taken equal to 1.0. The figure clearly shows that the value of L for which the Δ_3 statistic saturates moves up with dimension. The dimensional dependence is consistent with the scaling relation (2.13).

6. Conclusions

In this paper we have studied the Δ_3 statistic for systems with a potential given by a separable homogeneous polynomial. We have applied the analysis of Berry (1985) to this class of potentials and found that the Δ_3 statistic is reproduced by the analytical

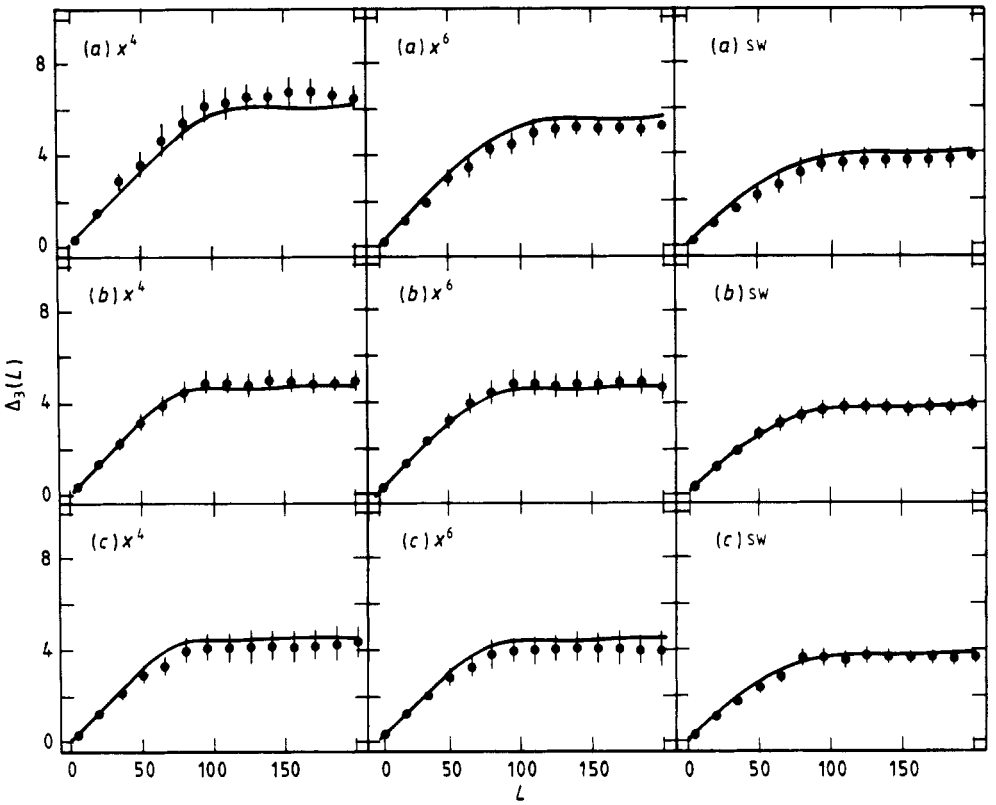

Figure 2. Comparison of the Δ_3 statistic calculated from the WKB eigenvalues and the analytical expression given in (4.4) for the 4500th eigenvalue up to the 5500th eigenvalue. The parameters of the potential are the same as in the corresponding part of figure 1. For further explanation see figure 1.

results. The dimensional and energy dependence of the Δ_3 statistic could be derived by invoking only the scaling properties of the potential. The results thus obtained were confirmed by the detailed analysis. In order to be able to unfold the spectra given by the WKB approximation (which was found to be excellent) accurately, we had to take into account a generalised perimeter term. This term appeared to have important contributions when the value of the zero-point fluctuations was different from 0.5. After taking into account this term the values of the Δ_3 statistic for the WKB eigenvalues could be reproduced satisfactorily. The dimensional dependence of the 'kink' in the Δ_3 statistic behaved according to a simple scaling relation. In higher dimensions the 'kink' moves to larger values of L. The functional dependence of the universal part of $\Delta_3(L)$ could be explained by scaling arguments both for the integrable and the ergodic cases.

A number of practical consequences ensue from these results. First, the 'kink' is irrelevant in nuclear statistics due to the large dimensionality of the system. Second, the 'kink' constitutes a deviation from Poisson statistics ubiquitous for integrable systems, though its location shifts along the spectrum and also depends on the Hamiltonian. Third, the exact availability of the long-range behaviour of the Δ_3 statistic for some systems may allow us to extract information from short low-lying spectra.

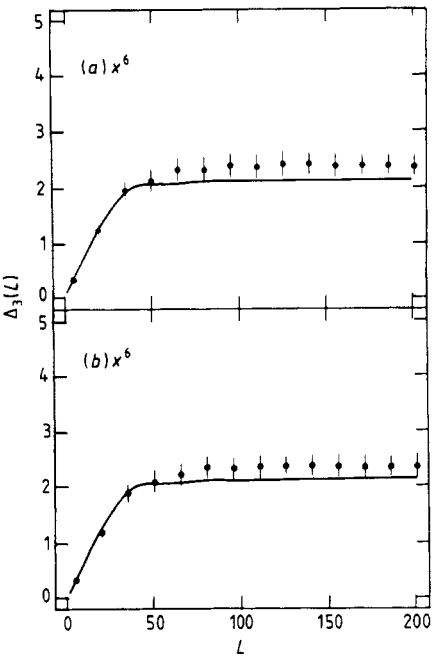

Figure 3. The Δ_3 statistic of the 500th eigenvalue up the 1500th WKB eigenvalue obtained without taking into account the perimeter term. The ratio of γ_1/γ_2 is equal to π. In (a) and (b) the values of the α_i are equal to (0.25, 0.25) and (0.25, 0.75), respectively. For further explanation see figure 1.

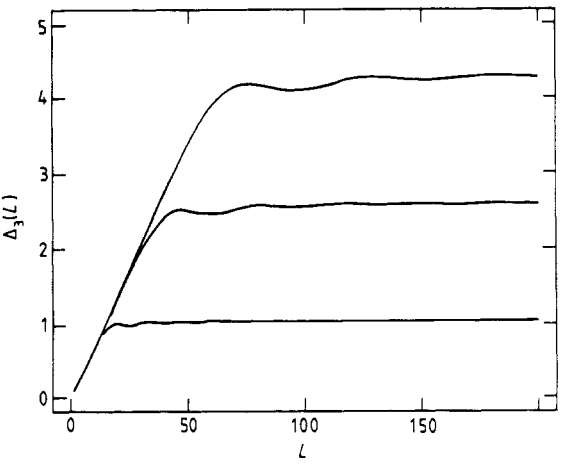

Figure 4. The dimensional dependence of the Δ_3 statistic. The full curves show the analytical expression (4.4) calulated for $d = 2$, 3 and 4 (going from the bottom to the top). The results have been computed for the 100th eigenvalue and the value of γ_i is equal to 1.0.

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